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NEW DEMONSTRATION OF THE BINOMIAL THEOREM.

BY JOHN MACNIE, A. M., EASTON, CONN.

1. Let the expansion of $(1+z)^n$ be

$$(1+z)^n = c_0 + c_1z + c_2z^2 + c_3z^3 + \dots . c_rz^r, \qquad [1]$$

where n may have any value, and $c_0, c_1, \dots c_r$, are coefficients independent of z.

Multiplying both members of [1] by $(1+z)$ we obtain

$$(1+z)^{n+1} = c_0+(c_1+c_0)z+(c_2+c_1)z^2+(c_3+c_2)z^3+ \dots (c_r+c_{r-1})z^r,$$

in which the $(r+1)$th coefficient is the sum of the $(r+1)$th and rth of [1].

By proceeding in the same manner, we obtain in succession,

$$(1+z)^{n+2} = c_0+(c_1+2c_0)z+(c_2+2c_1+c_0)z^2+(c_3+2c_2+c_1)z^3+ \dots$$
$$+(c_r+2c_{r-1}+c_{r-2})z^r,$$
$$(1+z)^{n+3} = c_0+(c_1+3c_0)z+(c_2+3c_1+3c_0)z^2+(c_3+3c_2+3c_1+c_0)z^3$$
$$+ \dots +(c_r+3c_{r-1}+3c_{r-2}+3c_{r-3})z^r$$
$$(1+z)^{n+4} = c_0+(c_1+4c_0)z+(c_2+4c_1+6c_0)z^2+(c_3+4c_2+6c_1+4c_0)z^3,$$
$$+(c_4+4c_3+6c_2+4c_1+c_0)z^4+ \dots +(c_r+4c_{r-1}+6c_{r-2}+4c_{r-3}+c_{r-4})z^r.$$

By noting the law according to which these coefficients are formed, we easily prove by induction that the $(r+1)$th term of $(1+z)^{n+m}$ is

$$\left[c_r+mc_{r-1}+\frac{m(m-1)}{2}c_{r-2}+ \dots mc_{r-m+1}+c_{r-m}\right]z^r,$$

and $$(1+z)^{n+m} = c_0+(c_1+mc_0)z+\left[c_2+mc_1+\frac{m(m-1)}{2}\right]z^2+ \dots \qquad [2]$$

If now in [1] we suppose $n = 0$, then since $(1+z)^0 = 1+0 \dots .$, we have $c_0 = 1$, $c_1 \dots . c_r$, each $= 0$, and we obtain the expansion of $(1+z)^m$ either from [2], or by substituting in succession, 0, 1, 2, ... m for r in the general term; noting that every term in the coeff. of that expression vanishes, except that for which the subscript of c becomes 0 for assumed value of r. Thus we obtain

$$(1+z)^m = 1+mz+\frac{m(m-1)}{1.2}z^2+\frac{m(m-1)(m-2)}{1.2.3}z^3+ \dots mz^{m-1}+mz^m,$$

m being any positive integer.

2. As by the aid of [2], the expansion of $(1+z)^n$ being given, we can find that of $(1+z)^{n+m}$, so, conversely, the expansion of $(1+z)^{n+m}$ being given, say

$$(1+z)^{n+m} = 1 + C_1z + C_2z^2 + C_3z^3 + \dots + C_rz^r, \qquad [4]$$

we can find in succession $c_1, c_2, c_3, \dots . c_r$, the coefficients of $(1+z)^n$, by means of the series of equations:

$$c_0 = 1;\ c_1 = C_1 - m;\ c_2 = C_2 - mc_1 - \frac{m(m-1)}{1.2};$$

$$c_3 = C_3 - mc_2 - \frac{m(m-1)}{1.2}c_1 - \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3};\ \&c.,$$

and thus obtain

$$(1+z)^n = 1 + (C_1 - m)z + \left(C_2 - mc_1 - \frac{m(m-1)}{1.2}\right)z^2$$

$$+\left(C_3 - mc_2 - \frac{m(m-1)}{1.2}c_1 - \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}\right)z^3. \qquad [5]$$

Since, by hypothesis, n may have any value, let $n = -m$, then [4] becomes $(1+z)^0 = 1+0+\ldots$, that is, $C_1 \ldots C_r$ each $= 0$; and substit'ng these values in [5], we obtain, after some simplifications:

$$(1+z)^{-m} = 1 + (-m)z + \frac{-m(-m-1)}{1.2}z^2 + \frac{-m(-m-1)(-m-2)}{1.2.3}z^3 + \ldots$$

where the exponent is any negative integer.

3. Again, since $(1+z)^{nm} = (1+z)^{n+(m-1)n}$, we can, if the expansion of $(1+z)^n$ is given, obtain that of $(1+z)^{nm}$ by substituting $(m-1)n$ for m in [2]; and, conversely, if the expansion of $(1+z)^{nm}$ be given, we can obtain that of $(1+z)^n$ by substituting $(m-1)n$ for m in [5].

Let $n = s \div m$, s and m being integers, then $(1+z)^{nm} = (1+z)^s$, and we have, by [3],

$C_1 = s$, $C_2 = \frac{s(s-1)}{1\,.\,2} \ldots$ Substituting, in [5], these values for $C_1 \ldots C_r$, and $(m-1)(s \div m)$ for m, and simplifying, we obtain

$$(1+z)^{\frac{s}{m}} = 1 + \frac{s}{m}z + \frac{(s\div m)[(s\div m)-1]}{1\,.\,2}z^2 + \frac{(s\div m)[(s\div m)-1][(s\div m)-2]}{1\,.\,2\,.\,3}z^3$$

$$+ \ldots\ldots$$

Hence, since s may be pos. or neg., we find, whatever n may be,

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{1\,.\,2}z^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}z^3 + \ldots$$

and thence, by putting $y \div x$ for z, and multiplying by x^n,

$$(x+y)^n = x^n + nx^{n-1}y + \ldots\ldots$$

SOLUTION OF PROB. 315 BY CHAS. H. KUMMELL. (SEE P. 166.)—The total distance of A is $D_a = 1+2+3 = 6^{\text{in}}$. It is required to find the most probable similar quantity for B.

The most probable position of B's bull's eye is evidently the center of gravity of his shots, viz., that point for which

$$D_1^2 + D_2^2 + D_3^2 = (x-0)^2 + (y-0)^2 + (x-1)^2 + (y-1)^2 + (x-2)^2 + (y-3)^2 = \text{min'm}.$$